P H Y S I C A L   R E V I E W   L E T T E R S

Boundary Multifractality at the Integer Quantum Hall Plateau Transition: Implications for the Critical Theory

H. Obuse,[1,*] A. R. Subramaniam,[2] A. Furusaki,[1] I. A. Gruzberg,[2] and A. W. W. Ludwig[3]

[1]*Condensed Matter Theory Laboratory, RIKEN, Wako, Saitama 351-0198, Japan*
[2]*James Franck Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637, USA*
[3]*Department of Physics, University of California, Santa Barbara, California 93106, USA*

We study multifractal spectra of critical wave functions at the integer quantum Hall plateau transition using the Chalker-Coddington network model. Our numerical results provide important new constraints which any critical theory for the transition will have to satisfy. We find a nonparabolic multifractal spectrum and determine the ratio of boundary to bulk multifractal exponents. Our results rule out an exactly parabolic spectrum that has been the centerpiece in a number of proposals for critical field theories of the transition. In addition, we demonstrate analytically exact parabolicity of the related boundary spectra in the two-dimensional chiral orthogonal "Gade-Wegner" symmetry class.

The physics of the quantum Hall effect has been an exciting area of research for more than two decades [1,2]. While much progress has been made in this area, the identification of an analytically tractable theory describing the critical properties at the transitions between the plateaus in the integer quantum Hall (IQH) effect has been elusive ever since [3]. These quantum phase transitions are famous examples of (Anderson) localization-delocalization (LD) transitions driven by disorder. The diverging localization length plays the role of a correlation length in nonrandom continuous phase transitions, known to be described by conformal field theories in two dimensions (2D). It is natural to expect that effective (field) theories describing IQH plateau transitions should generally also possess conformal symmetry (cf. [4]).

Many attempts have been made in the past to identify an analytically tractable description of the IQH plateau transition, and, more recently, Wess-Zumino (WZ) field theories defined on a certain supermanifold were conjectured to provide such a description [5–7]. (Similar theories have also appeared in the context of string propagation in anti-de Sitter space-time [8].) These proposals focused solely on bulk observables, i.e., on physical quantities measured in a sample without any boundaries. In this Letter, we provide important new constraints that arise when one studies the scaling behavior of wave functions near the boundaries of a sample. Any proposed candidate theory for the plateau transitions will have to be consistent with our numerical results for the boundary multifractal spectrum.

At LD transitions, critical wave functions obey scale-invariant, multifractal (MF) statistics; namely, disorder-averaged moments of wave functions have a power-law dependence on the linear dimension L of the system [9]:

$$\overline{|\psi(r)|^{2q}}/(\overline{|\psi(r)|^2})^q = C_q^x(L)L^{-\Delta_q^x}. \tag{1}$$

The MF exponents Δ_q^x, which are related to ("anomalous") scaling dimensions of certain operators in an underlying

field theory [10], can be defined for points r in the bulk ($x = b$) of the sample Δ_q^b or near its boundary ("surface": $x = s$) [4,11] Δ_q^s. The prefactor $C_q^x(L)$ in Eq. (1) depends on q and, in general, on L if we include the possibility of corrections to scaling. Both sets of MF exponents satisfy the symmetry relation [12]

$$\Delta_q^x = \Delta_{1-q}^x \tag{2}$$

(in some interval [4] around $q = 1/2$).

Equivalently, the MF wave functions can be characterized by the so-called singularity spectra $f^x(\alpha^x)$ related to Δ_q^x by a Legendre transform: $f^x(\alpha_q^x) = (\alpha_q^x - 2)q - \Delta_q^x + d_x$, $\alpha_q^x - 2 = d\Delta_q^x/dq$, and $d_b = 2$, $d_s = 1$. The exponent α_0^x describes the scaling of typical wave functions: $\overline{\ln|\psi(r)|^2} \sim -\alpha_0^x \ln L$, as can be seen by taking the q derivative in Eq. (1) at $q = 0$.

Work emerging [13,14] from Ref. [5] led to the conjecture that the proposed theory would give rise to an exactly parabolic bulk MF spectrum for the IQH transition

$$\Delta_q^b = \gamma^b q(1 - q), \tag{3}$$

reminiscent of analytically obtained MF spectra for Dirac fermions in, e.g., random Abelian gauge potentials [15,16]. In those models, the parabolicity of the MF spectrum can be understood through a reformulation of the problem in terms of free fields.

Previous numerical studies [13] of wave function statistics at the IQH transition appeared to exhibit a bulk MF spectrum that was indeed well described (with an accuracy of ~1%) by a parabolic fit (3) with $\gamma^b = 0.262 \pm 0.003$, seemingly providing support for the conjectures advanced in Refs. [5–7]. [In Ref. [13], the results are presented in terms of $f^b(\alpha)$. For a parabolic MF spectrum (3), $f^b(\alpha^b)$ is also parabolic, with a maximum at $\alpha_0^b = \gamma^b + 2$.]

Besides its conjectured relevance [5] to the IQH transition, the above-mentioned WZ theory is known to de-

scribe transport properties of a disordered electronic system in a different universality class [17,18] (the chiral unitary Gade-Wegner class AIII of Refs. [19,20]) which possesses an additional discrete (chiral) symmetry [19], not present in microscopic models for the IQH transition. Well-known microscopic realizations of field theories in class AIII are random bipartite hopping models and certain network models [17,18,21]. The theory possesses a line of fixed points, with continuously varying critical properties parametrized by the critical longitudinal DC conductivity. (It was argued in Ref. [5] that, for a particular value of this continuous parameter, the WZ theory would provide a description of the IQH transition.)

In this Letter, we obtain two kinds of results. First, we provide results of extensive numerical work on the MF exponents at the IQH transition both at a boundary (Δ_q^s) and in the bulk (Δ_q^b). Based on these numerical results, quadratic behavior in q is ruled out for both quantities. Deviations from the parabolic form (3) are found to be much larger in the MF exponents Δ_q^s at a boundary. Here it is important to note that, in complete analogy to the bulk, the above conjectures would *also* yield a quadratic dependence on q of the *boundary* MF exponents Δ_q^s. We further determine the ratio Δ_q^s/Δ_q^b over a range of q. Accounting for this ratio is an important constraint on any proposed critical theory for the transition.

Second, we demonstrate analytically the exact parabolicity of boundary spectra, not for the chiral unitary class AIII, but for the related time-reversal invariant version, the chiral orthogonal Gade-Wegner class BDI [17–19,21]. We expect such parabolicity to also hold in the chiral unitary symmetry class.

We begin with the numerical part. Here we study the multifractality of critical wave functions in a way similar to Ref. [13], with the goal of numerically determining the rescaled anomalous exponents

$$\gamma_q^x = \Delta_q^x/q(1-q), \qquad (4)$$

for both $x = s$ (boundary) and $x = b$ (bulk).

For the case of boundary exponents, we consider the critical Chalker-Coddington network model (CCNM) [22] with $4L^2$ links placed on a cylinder. The dynamics of wave functions on links of the network is governed by a unitary evolution operator U. For each disorder realization, we numerically diagonalize U and retain one critical wave function whose eigenvalue is closest to 1. The largest system size we studied was $L = 180$, and the ensemble average was taken over 3×10^5 samples for $L = 50, 60$, 5×10^5 samples for $L = 80$, and 2×10^5 samples for $L = 120, 180$.

We obtain the anomalous dimensions Δ_q^s from Eq. (1). The boundary wave function coarse-grained over each plaquette along the boundary is substituted into the left-hand side of Eq. (1), where the overline denotes ensemble and spatial averages along the boundary. Taking the logarithm, we numerically obtain $D_q^s(L) \equiv (q \ln\overline{|\psi(r)|^2} -$

$\ln\overline{|\psi(r)|^{2q}})/\ln L = \Delta_q^s - \ln C_q^s(L)/\ln L$ and plot this quantity as a function of $1/\ln L$ in Fig. 1. We see that corrections to scaling are significant for small systems ($L \lesssim 30$). Therefore, we used our numerical data for $L \gtrsim 50$ only to extract γ_q^s by linear fitting.

Independently, we numerically obtain α_q^s and f^s from $\overline{|\psi(\mathbf{r})|^{2q} \ln|\psi(\mathbf{r})|^2}/\overline{|\psi(\mathbf{r})|^{2q}} \sim -\alpha_q^s \ln L$ and $\ln\overline{|\psi(\mathbf{r})|^{2q}} \sim [f^s(\alpha_q^s) - \alpha_q^s q - d_s]\ln L$, using our numerical data for $L \gtrsim 50$. For example, the exponent α_0^s is obtained by linear fitting (Fig. 1), $A^s \equiv -\overline{\ln|\psi(r)|^2}/\ln L = \alpha_0^s + \text{const}/\ln L$, which yields $\alpha_0^s = 2.386 \pm 0.004$.

We show in Fig. 2(a) the rescaled boundary anomalous dimension γ_q^s (solid red circles) obtained from this analysis. We see clearly that γ_q^s is not constant, implying that the boundary MF spectrum Δ_q^s is *not parabolic*. The change in γ_q^s over the interval $0 < q \leq 1/2$ is about 4%–5% and is significantly larger than the error bars. This provides the strongest numerical evidence against the parabolicity of the MF exponents.

Shown in the same figure by open blue circles is the mirror image of γ_q^s with respect to $q = 1/2$, γ_{1-q}^s. We see that the symmetry relation (2) is satisfied within error bars for $0 \lesssim q \lesssim 1$. The rescaled anomalous dimension γ_q^s approaches $\alpha_0^s - 2$ (the horizontal line) at $q = 0, 1$, indicating that the two independent calculations of α_0^s and Δ_q^s are consistent.

We have also computed the bulk anomalous dimension Δ_q^b using the CCNM on a torus. In this case, the overline in Eq. (1) implies both the ensemble and the spatial average over the whole torus. Wave functions are coarse-grained on each plaquette. We have employed the same fitting procedure as in the boundary case. The biggest system size we examined for the bulk analysis is $L = 270$. The number of samples over which we took the average is 5×10^5 for $L = 50$, 3×10^5 for $L = 60, 80$, 2×10^5 for $L = 120$, 4×10^4 for $L = 180$, and 2×10^4 for $L = 270$.
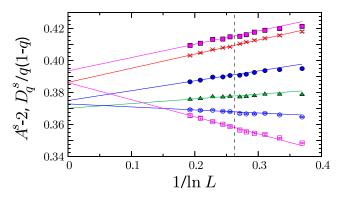

FIG. 1 (color online). System size dependence of $D_q^s(L)/q(1-q)$ defined in the main text for $q = -0.2$ (■), 0.2 (●), 0.5 (▲), 0.8 (○), and 1.2 (□). γ_q^s is calculated by linear fitting taking into account only for larger system sizes ($L \geq 50$) indicated by the vertical dashed line between $L = 40$ and 50. We also show $A^s - 2$ defined in the main text (×).
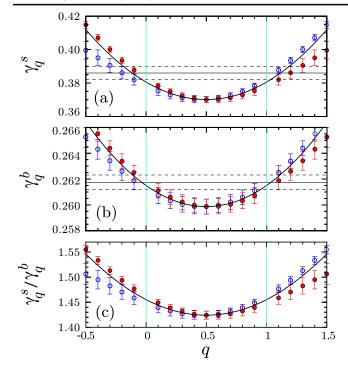
FIG. 2 (color online).   (a) Rescaled boundary MF exponents γ_q^s (●) and γ_{1-q}^s (○). The curve is $0.370 + 0.042(q - 1/2)^2$, obtained by fitting the data for γ_q^s in $0 < q < 1$ to a parabolic form. The horizontal solid line shows $\alpha_0^s - 2 = 0.386 \pm 0.004$ with error bars indicated by dashed lines, which is consistent with $\lim_{q \to 0,1} \gamma_q^s$. (b) Rescaled bulk MF exponents γ_q^b (●) and γ_{1-q}^b (○). The curve is $0.2599 + 0.0065(q - 1/2)^2$, obtained by fitting the data for γ_q^b in $0 < q < 1$ to a parabolic form. The horizontal solid line shows $\alpha_0^b - 2 = 0.2617 \pm 0.0006$ with error bars indicated by dashed lines. (c) Ratios γ_q^s/γ_q^b (●) and $\gamma_{1-q}^s/\gamma_{1-q}^b$ (○). As above, the curve is obtained from the parabolic fits for $\gamma_q^{s,b}$, which amounts to quartic approximations for $\Delta_q^{s,b}$.

Figure 2(b) shows the exponents γ_q^b, together with their mirror image. The symmetry relation (2) is again satisfied for $0 \lesssim q \lesssim 1$ within error bars, which provides confirmation that our results are reliable. We see clearly that γ_q^b has q dependence, although it is weaker than that of γ_q^s; compare the vertical scales of Figs. 2(a) and 2(b).

The ratio γ_q^s/γ_q^b is shown in Fig. 2(c) and is seen to be clearly dependent on q. Any candidate theory for the IQH transition needs to be consistent with this ratio and, in particular, its dependence on q. (Note that for a free field this ratio would be equal to 2 and independent of q [11,23].)

Figure 3(a) shows α_q^x as a function of q. The data significantly deviate from linear behavior that would result if Δ_q^x were strictly parabolic (constant γ_q^x). In Fig. 3(b), we show the corresponding singularity spectra $f^x(\alpha_q^x)$ as functions of q. [Combining the data from the two panels would result in $f^x(\alpha^x)$ as functions of α^x.] For $q \gtrsim 1.5$ where $f^s(\alpha^s) < 0$, the moments $\overline{|\psi(r)|^{2q}}$ are dominated by rare events, and thus accurate numerical calculation of MF
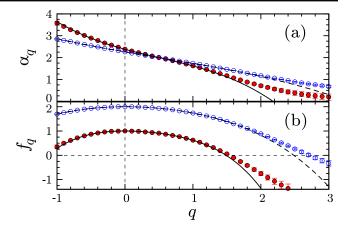

FIG. 3 (color online).   (a) α_q^s (●) and α_q^b (○) as functions of q; (b) f_q^s (●) and f_q^b (○) as functions of q. The solid and dashed curves on both panels are obtained from the parabolic approximations to γ_q^x (that is, quartic approximations to Δ_q^x). Notice that α_q^x significantly deviate from straight lines which would follow from strictly parabolic Δ_q^x (or constant γ_q^x).

exponents becomes more difficult [9]. This explains the discrepancy between the (red) data points and the solid curves for $q \gtrsim 1.5$ in Fig. 3. As $f_q^s > 0$ at $q \gtrsim -1$, we expect that our numerical results of $f^s(\alpha^s)$ should be more reliable at $q \approx -1$ than at $q \approx 1.5$, as evidenced by the agreement between the red dots and the solid curve in Fig. 3. The curve suggests termination of $f^s(\alpha^s)$ [9] to occur at $q \approx 2.2$.

We now proceed to present our second (analytical) result. We first recall that the theory conjectured in Ref. [5] to describe the IQH plateau transition is a WZ model with global psl(2|2) (super)symmetry. This theory possesses two coupling constants: one denoted by f for the kinetic term and another denoted by k for the WZ term ($k = 1$ in Ref. [5]), in standard notation [5,24]. One can think of this theory as a perturbation of the rather well understood [24,25] Kac-Moody (KM) point characterized by the condition $f^{-2} = k$, perturbed by a term in the action of the form [24,26] $\delta S = (\lambda/k^2) \int d^2z \phi_{ab}(z, \bar{z}) J^a(z) \bar{J}^b(\bar{z})$, where $\lambda = f^{-2} - k$. Here ϕ_{ab} is the KM primary field in the adjoint representation of psl(2|2), J^a and $\bar{J}^b$ are the left and right chiral components of the psl(2|2) Noether currents [24], respectively, and λ parametrizes the line of fixed points, mentioned above.

The conjectured link [5] between the WZ model and the IQH transition can be formulated through the notion of the point contact conductance (PCC) [27]. The PCC is a statistically fluctuating quantity. (i) On one hand, the scaling dimension X_q of the qth moment of the PCC at the IQH transition has been *proven* to be simply related to the exponent Δ_q [13,14], $X_q = 2\Delta_q$ ($|q| < 1/2$). (ii) On the other hand, the scaling dimension x_q of the operator *in the WZ model* carrying the same representation of the global psl(2|2) symmetry as the qth moment of the PCC in the CCNM (possessing the same psl(2|2) symmetry) was *conjectured* [5] to be a quadratic function of q. (iii) If one

combines (i) and (ii), and if one assumes $X_q = x_q$ (following the conjectured description of the IQH transition by the WZ model), then the wave function exponents Δ_q *at the IQH transition* would be quadratic functions of q, as in Eq. (3).

As already mentioned, this WZ theory is known to describe transport properties of the chiral unitary class AIII [17,18,21], lacking time-reversal symmetry. Below, we demonstrate the correctness of the conjecture made in item (ii) above, at a *boundary*, and for the *time-reversal invariant version* of the AIII model, the chiral orthogonal class BDI [17,19]. Just as its cousin with broken time-reversal symmetry, the chiral orthogonal theory also possesses a line of fixed points. Transport properties along this line can be described [18] by the perturbation of the KM point of the psl(2|2)-invariant WZ theory described above, when the field ϕ_{ab} is replaced by the Kronecker delta $\phi_{ab} \rightarrow \delta_{ab}$. Denote the corresponding coupling constant by λ_t. Consider the theory in the upper half plane, where the system simply ends at the boundary, and an operator of scaling dimension (a ''conformal weight'') $x_\rho^s(\lambda_t)$ on the boundary. At the KM point, where the perturbation vanishes, $\lambda_t = 0$, and such an operator is described by a representation ρ of the global psl(2|2) symmetry. It is known [24] that $x_\rho^s(\lambda_t = 0) = C_\rho^{(2)}/k$, where $C_\rho^{(2)}$ is the quadratic Casimir invariant in the representation ρ. It turns out to be straightforward [28] to compute the change of the scaling dimension, order by order in the bulk coupling constant λ_t, yielding a geometric series. The result is simply $x_\rho^s(\lambda_t) = C_\rho^{(2)}/(k + \lambda_t)$. Note that, for the (continuous series) representation ρ of psl(2|2) in which the qth moment of the PCC at the IQH transition transforms, one has $C_\rho^{(2)} = q(1 - q)$. This proves our claim that the spectrum of scaling dimensions $x_\rho^s \rightarrow x_q^s$ of corresponding boundary operators in symmetry class BDI is a strictly quadratic function of q.

In summary, our numerical results clearly demonstrate that both the boundary and the bulk MF spectra Δ_q^s and Δ_q^b significantly deviate from parabolicity and that their q-dependent ratio is significantly different from 2. (These conclusions were recently also reached, independently, by Evers, Mildenberger, and Mirlin [29].) These results for the bulk as well as the boundary MF spectra impose important constraints on any analytical theory for the IQH plateau transition. Furthermore, we have demonstrated analytically exact parabolicity of related boundary spectra in the 2D chiral orthogonal Gade-Wegner symmetry class BDI.

[*]Present address: Department of Physics, Kyoto University, Sakyo-ku, Kyoto 606-8502, Japan.

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